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Jacobi fields and linear connections for arbitrary second-order ODEs

M. Jerie [a,*], G.E. Prince [b,1]

[a] *Department of Mathematics, La Trobe University, Vic. 3086, Australia*
[b] *School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia*

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Abstract

We discuss generalisations of Jacobi fields and Raychaudhuri's equation from the geodesic case to that of an arbitrary system of second-order ODEs. Our results are obtained using a natural choice of linear connection on evolution space.

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1. Introduction

In this work, we aim to generalise parts of the geometry of geodesics on a Riemannian manifold with linear connection to the solutions of an arbitrary system of second-order ordinary differential equations. In particular, we give a generalisation of the geodesic deviation equation, and Raychaudhuri's equation.

We represent a system of equations

$$\ddot{x}^a = f^a(t, x, \dot{x})$$

on a configuration manifold M by a vector field Γ (called a second-order differential equation field or SODE) on the *evolution space* $E := \mathbb{R} \times TM$. Our generalisation then uses two

* Corresponding author.
E-mail addresses: m.jerie@latrobe.edu.au (M. Jerie), g.prince@latrobe.edu.au (G.E. Prince).
[1] Present address: Department of Mathematics, La Trobe University, Vic. 3086, Australia.

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constructions from Γ, both on E: firstly a linear connection (with torsion), $\hat{\nabla}$, and secondly a type $(1, 1)$ tensor field, A_Γ, which describes the deformation of the tangent spaces of E due to the flow generated by Γ. This second object can be defined on any differentiable manifold N with *linear* connection ∇, associated torsion T and local vector field Z. Denoting parallel transport using ∇ on N by τ_t and denoting the flow generated by Z by ζ_t, we have the following definition.

Definition 1.1.

$$A_Z(\xi) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_t^{-1}(\zeta_{t*}\xi), \quad \text{where } \xi \in T_x N.$$

Maps defined in this way are examples of what we will call *shape maps*: as we will see it is not always necessary to have a linear connection in order to define such maps. The simplest possible case is that of $\mathbb{R}^2$ with the Euclidean metric and flat connection: the corresponding map can be used in the geometric analysis of planar flows (see [13]).

Theorem 1.2.

$$A_Z(\xi) = \nabla_\xi Z + T(Z_x, \xi), \quad \xi \in T_x N.$$

Proof. Let X be the field obtained by Lie dragging ξ along the integral curve of Z through x. Then

$$\begin{aligned} A_Z(\xi) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\tau_t^{-1} X_{\zeta_t(x)}) = (\nabla_Z X)_x = (\nabla_X Z)_x + T(Z, X)_x + (\mathcal{L}_Z X)_x \\ &= \nabla_\xi Z + T(Z_x, \xi), \end{aligned}$$

where we have used $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ and $\mathcal{L}_Z X = 0$. □

When the connection is symmetric A_Z is just ∇Z, the covariant differential of Z. The generalised Raychaudhuri equation is obtained by assuming Z is geodesic with respect to ∇ and taking the trace of $\mathcal{L}_Z A_Z$. Vector fields satisfying $A_Z X = \nabla_Z X$ along a geodesic field Z can be shown to satisfy a generalised Jacobi's equation. We will elaborate on these ideas and apply them to $\hat{\nabla}$ and A_Γ.

The present paper is a development of two earlier ones. In [3], Crampin and Prince discussed the conventional case of the Levi-Civita connection and related Raychaudhuri's equation on the configuration space M to a universal equation on TM involving the geodesic spray Γ. They introduced the map A_Z on TM but they did not, however, introduce a linear connection or a map A_Γ. In a second paper [7], Jerie and Prince generalised the results of Crampin and Prince [3] to an arbitrary system of second-order ODEs by introducing a map A_Z (this time on $T(\mathbb{R} \times M)$) in the absence of a linear connection. We demonstrated that our generalised Raychaudhuri equation does describe caustics and singularities of congruences of solution curves in an exactly analogous manner to its traditional counterpart, so effectively used in general relativity (see, for example, the books [6,17]). In the current paper, we give an alternative derivation of the generalised Raychaudhuri equation on $\mathbb{R} \times M$ and revisit the Jacobi field idea, but most importantly we introduce a linear connection, $\hat{\nabla}$ and the map A_Γ on E, develop both Raychaudhuri and Jacobi equations on E and relate these to our earlier work.

This paper is set out as follows. In Section 2 we give a brief account of the Riemannian setting. In Section 3 we introduce evolution space E, and show how the ideas of Section 2 apply to the solutions of an arbitrary system of SODE. In Section 4 we describe a linear connection $\hat{\nabla}$ on E, and in Section 5 we show how A_Γ and the linear connection combine to give a more satisfactory global version of the SODE case treated in Section 3. Section 6 discusses the Raychaudhuri and Jacobi equations associated with A_Γ. Appendix A deals with the tensorial character of A_Z.

2. The Riemannian setting: Raychaudhuri's equation and Jacobi tensors

For the sake of completeness we very briefly describe the relevant parts of Crampin and Prince [3] (our first paper [7] gives a more self-contained account). The setting is an n-dimensional smooth manifold M equipped with a metric g and its symmetric, linear connection ∇. Define a type (1, 1) tensor field A_Z associated with a local vector field Z on M by comparing Lie transport with parallel transport:

$$A_Z := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_t^{-1} \circ \zeta_{t*}. \tag{2.1}$$

Here ζ_t is the flow generated by Z, and τ_t the parallel transport map along ζ_t. The trace of A_Z is a measure of the divergence of ζ_t. Raychaudhuri's equation describes the evolution of this trace along the flow when Z is geodesic. For any ξ tangent to M it is simple to show that $A_Z(\xi) = \nabla_\xi Z$, and if Z is geodesic (more correctly auto-parallel) then the propagation equation for A_Z along Z is

$$\mathcal{L}_Z A_Z = \nabla_Z A_Z = -R_Z - A_Z^2, \tag{2.2}$$

where R_Z is the type (1, 1) tensor field obtained from the Riemann curvature, Riemann, of the connection by

$$R_Z(X) := \mathrm{Riemann}(X, Z)Z.$$

Since the operations of taking the trace and Lie differentiation commute, the trace of Eq. (2.2) is Raychaudhuri's equation.

The propagation equation (2.2) is also the key to a simple derivation of the Jacobi equation and its extension to tensor fields. Any Lie-dragged vector field X along an integral curve ζ of Z satisfies $\nabla_{\dot\zeta} X = A_Z(X)$ and one immediately has

$$\nabla_{\dot\zeta}^2 X = (\nabla_Z A_Z)(X) + A_Z^2(X) = -R_Z(X).$$

Similarly, any tensor field J along ζ with $\nabla_{\dot\zeta} J = A_Z \circ J$ satisfies the Jacobi tensor field equation

$$\nabla_{\dot\zeta}^2 J + R_Z \circ J = 0.$$

3. The SODE setting: Raychaudhuri's equation and Jacobi tensors

In what follows, we will be analysing a system of second-order differential equations

$$\ddot{x}^a = f^a(t, x, \dot{x}) \tag{3.1}$$

on a manifold M with local coordinates (x^a) and with associated bundles $\pi : \mathbb{R}\times M \to M$, $t : \mathbb{R}\times M \to \mathbb{R}$ and $\pi_1^0 : E \to \mathbb{R}\times M$. The geodesic equations are, of course, a special example and one might expect the analysis to be modelled on that situation. However, we take the position that even the autonomous case is best described on *extended configuration space* $\mathbb{R}\times M$ and evolution space $E := \mathbb{R}\times TM$ and that one should put aside the fact that historically autonomous systems were discussed on M and its tangent and cotangent bundles. While the paper [3] did not use this more general setting, we refer the reader to the papers [11,12] for a fully nonautonomous treatment of the geodesic case (in the context of projective differential geometry). The papers [4,7] are basic references for the material in this section.

From (3.1) we construct on E (with local, adapted coordinates (t, x^a, u^a)) an SODE:

$$\Gamma = \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + f^a\frac{\partial}{\partial u^a}, \tag{3.2}$$

whose integral curves are the 1-jets of the solution curves of the given equations.

The vertical and contact structures of the bundle $\pi_1^0 : E \to \mathbb{R}\times M$ are combined in S, an intrinsic $(1, 1)$ tensor field on E and known as the *vertical endomorphism*. In coordinates:

$$S = V_a \otimes \theta^a, \tag{3.3}$$

where $V_a := \partial/\partial u^a$ are the vertical basis fields and $\theta^a := \mathrm{d}x^a - u^a\,\mathrm{d}t$ are the local contact forms. From the first-order deformation, $\mathcal{L}_\Gamma S$, a *nonlinear connection* is constructed as follows: $\mathcal{L}_\Gamma S$ has eigenvalues $0, 1, -1$ with corresponding eigenspaces spanned locally by Γ, the n *vertical* fields V_a and n *horizontal* fields

$$H_a := \frac{\partial}{\partial x^a} - \Gamma_a^b\frac{\partial}{\partial u^a}, \quad \text{where } \Gamma_a^b := -\frac{1}{2}\frac{\partial f^a}{\partial u^b}, \tag{3.4}$$

respectively.

The vector fields $\{\Gamma, H_a, V_a\}$ form a local basis on E, with dual basis $\{\mathrm{d}t, \theta^a, \psi^a\}$, where

$$\psi^a := \mathrm{d}u^a - f^a\,\mathrm{d}t + \Gamma_b^a\theta^b.$$

The Γ_a^b form the components of the nonlinear connection thus induced by Γ. The resulting direct sum decomposition of $T(E)$ is $I_E = P_\Gamma + P_H + P_V$, where I_E is the identity type $(1, 1)$ tensor field on E, and P_Γ, P_H and P_V are the three projection operators given in coordinates by

$$P_\Gamma = \Gamma\otimes\mathrm{d}t, \qquad P_H = H_a\otimes\theta^a, \qquad P_V = V_a\otimes\psi^a. \tag{3.5}$$

(In our earlier paper, we used N, P and Q for these projectors: we have changed our notation for the purpose of comparison in the next section with [2].) The components of the Jacobi endomorphism, $\Phi := P_V\circ\mathcal{L}_\Gamma P_H$, a type $(1, 1)$ tensor field on E, can be calculated from

$$[\Gamma, H_a] = \Gamma_a^b H_b + \Phi_a^b V_b, \tag{3.6}$$

giving

$$\Phi = \Phi^b_a V_b \otimes \theta^a = (B^b_a - \Gamma^b_c \Gamma^c_a - \Gamma(\Gamma^b_a)) V_b \otimes \theta^a, \tag{3.7}$$

where $B^b_a := -(\partial f^b/\partial x^a)$. Other useful results:

$$[\Gamma, V_a] = -H_a + \Gamma^b_a V_b, \qquad [H_a, H_b] = R^d_{ab} V_d, \tag{3.8}$$

this second fact is effectively the definition of the curvature, R, of the nonlinear connection Γ^b_a.

In [4], vertical and horizontal lifts to E of vector fields on $\mathbb{R} \times M$ are intrinsically defined; here it suffices to give their coordinate descriptions. Given $X \in \mathfrak{X}(\mathbb{R} \times M)$ with coordinate representation $X = X^0(\partial/\partial t) + X^a(\partial/\partial x^a)$ then

$$X^V = (X^a - u^a X^0) V_a, \qquad X^H = (X^a - u^a X^0) H_a.$$

This means, for example, that for any vertical vector $\mu \in T_q(E)$ there exists a unique vector $\eta \in T_{\pi^0_1(q)}(\mathbb{R} \times M)$ with $\mathrm{d}t(\eta) = 0$ such that $\eta^V = \mu$.

The following simple but important result is obtained using $\Phi := P_V \circ \mathcal{L}_\Gamma P_H$ and $I_E = P_\Gamma + P_H + P_V$.

Proposition 3.1.

$$P_V \circ \mathcal{L}_\Gamma P_V = -\Phi. \tag{3.9}$$

In order to arrive at a generalised Raychaudhuri equation for SODEs we need to introduce an arbitrary congruence of (graphs) of solution curves of (3.1). We follow [7]: assume the existence of such a congruence with corresponding local tangent field $Z \in \mathfrak{X}(\mathbb{R} \times M)$. Then, for local functions Z^a on $\mathbb{R} \times M$, we can write

$$Z = \frac{\partial}{\partial t} + Z^a \frac{\partial}{\partial x^a}.$$

The relation between Z and (3.1) is given by

$$Z(Z^a) = f^a(t, x^b, Z^b).$$

Z defines a local section, σ_Z, of $\pi^0_1 : E \to \mathbb{R} \times M$ by

$$\sigma_Z(p) := (p, \pi_* Z_p).$$

We will use an overline to indicate the restriction in E to the image of the section. At the risk of a mild ambiguity we will also use an overline to denote the pullback by the section, so that, for example, $\bar{\theta}^a := \mathrm{d}x^a - Z^a\,\mathrm{d}t$ denotes both the restriction and the pullback of the contact forms. We will also use the symbols $\overset{*}{=}$ and $:\overset{*}{=}$ for section equality and definition on the section, respectively. Then the fact that Z is tangent to graphs of solution curves of (3.1) is expressed as $\bar{f}^a = Z(Z^a)$ (as already noted) and to $\Gamma \overset{*}{=} \sigma_{Z*}(Z)$.

As we explained in [7] we can still define a map A_Z even though we do not have a linear connection on M. We do this by using the result that $A_Z = \sigma^*_Z P_V$ in the geodesic case. We give a brief summary. Pull P_V back from E to $\mathbb{R} \times M$ using the section: let $\xi \in T_p(\mathbb{R} \times M)$, then $P_V(\sigma_{Z*}\xi)$ is vertical, and hence there is a unique vector $\eta \in T_p(\mathbb{R} \times M)$ such that $\mathrm{d}t(\eta) = 0$ and $\eta^V = P_V(\sigma_{Z*}\xi)$. We denote the linear map $\xi \mapsto \eta$ by $\sigma_Z^* P_V$, hence

$$(\sigma_Z^* P_V(\xi))^V = P_V(\sigma_{Z*}\xi), \qquad \mathrm{d}t(\sigma_Z^* P_V) = 0. \tag{3.10}$$

This can be done for any vertical (1, 1) tensor field B on E (vertical means that $P_V \circ B = B$) to give $\sigma_Z^* B$, see [7]. In particular, $\Phi := P_V \circ \mathcal{L}_\Gamma P_H$ is vertical and we will denote $\sigma_Z^* \Phi$ by $\bar{\Phi}$ and its dual action on forms by $\bar{\Phi}^*$.

Definition 3.2. We define the type (1, 1) tensor field A_Z on $\mathbb{R} \times M$ associated with Z by

$$A_Z := \sigma_Z^* P_V. \tag{3.11}$$

We denote the dual action on 1-forms by A_Z^*, so that $A_Z^*(\omega) := \omega \circ A_Z$.

In coordinates

$$A_Z = \left(\frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}_b^a\right) \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b. \tag{3.12}$$

In [7], we show that

$$\mathcal{L}_Z A_Z = -A_Z^2 - \bar{\Phi}. \tag{3.13}$$

The trace of this equation is the generalisation of Raychaudhuri's equation. Importantly, we showed that Eq. (3.13) is the pullback by σ_Z of Eq. (3.9), this latter equation being a sort of universal evolution equation describing all congruences of graphs of solution curves.

The next object to be generalised, at least in part, is the covariant derivative itself. This derivative is used in [7] to prove that the zeros of $(\mathrm{trace}(A_Z))^{-1}$ determine congruence collapse.

Definition 3.3. We define a covariant derivative-like operator $\bar{\nabla}$ which acts only along Z to be the linear operator with the properties:

(i) $\bar{\nabla}(f) := Z(f)$ for all $f \in C^\infty(\mathbb{R} \times M)$,
(ii) $\bar{\nabla}(X) := [Z, X] + A_Z(X)$ for all $X \in \mathfrak{X}(\mathbb{R} \times M)$,
(iii) $(\bar{\nabla}\omega)(X) := \bar{\nabla}(\omega(X)) - \omega(\bar{\nabla}X)$ for all $\omega \in \mathfrak{X}^*(\mathbb{R} \times M)$,
(iv) $\bar{\nabla}$ acts by the Leibniz rule on tensor and wedge products and commutes with tensor contractions.

We remark that part (iii) of the definition means that on 1-forms $\bar{\nabla} = \mathcal{L}_Z - A_Z^*$.

In our earlier paper, we did not use $\bar{\nabla}$ or (3.9) in establishing Eq. (3.13) but with their help a much cleaner proof can be achieved which we now present.

Proposition 3.4. $\bar{\nabla} A_Z = \mathcal{L}_Z A_Z$ *and* $\bar{\nabla} A_Z^* = \mathcal{L}_Z A_Z^*$.

Proof. Recall from the definition that, acting on vector fields, one has $\bar{\nabla} = \mathcal{L}_Z + A_Z$. Then

$$\begin{aligned}\bar{\nabla} A_Z &= \bar{\nabla} \circ A_Z - A_Z \circ \bar{\nabla} = (\mathcal{L}_Z + A_Z) \circ A_Z - A_Z \circ (\mathcal{L}_Z + A_Z) \\ &= \mathcal{L}_Z \circ A_Z - A_Z \circ \mathcal{L}_Z = \mathcal{L}_Z A_Z.\end{aligned}$$

The same result holds for the action of A_Z^* on 1-forms. □

The following result could serve as part of an alternative definition of $\bar{\nabla}$.

Lemma 3.5. *Let* $X \in \mathfrak{X}(\mathbb{R} \times M)$ *such that* $\mathrm{d}t(X) = 0$, *i.e.* $X = X^a(\partial/\partial x^a)$, *then*

$$(\bar{\nabla} X)^V = P_V(\mathcal{L}_\Gamma(X^V)).$$

Proof. A straightforward coordinate calculation suffices. □

Lemma 3.6. *Let* $X \in \mathfrak{X}(\mathbb{R} \times M)$. *Then*

$$\bar{\Phi}(X) = -\bar{\nabla}(A_Z(X)) + A_Z([Z, X]).$$

Proof. By definition $\bar{\Phi} := \sigma_Z^* \Phi$, hence $\bar{\Phi}(X)^V = \Phi(\sigma_{Z*}X)$. Therefore, using Proposition 3.1,

$$\begin{aligned}\bar{\Phi}(X)^V &= -P_V \circ \mathcal{L}_\Gamma P_V(\sigma_{Z*}X) = -P_V(\mathcal{L}_\Gamma(A_Z X)^V) + P_V(\sigma_{Z*}(\mathcal{L}_Z X)) \\ &= -(\bar{\nabla}(A_Z X))^V + A_Z([Z, X])^V,\end{aligned}$$

where we have also used Lemma 3.5 in the last step. Hence

$$\bar{\Phi}(X) = -\bar{\nabla}(A_Z(X)) + A_Z([Z, X]).$$ □

Lemma 3.7.

$$\bar{\Phi}^* = -\bar{\nabla} \circ A_Z^* + A_Z^* \circ \mathcal{L}_Z - 2A_Z^{*2}.$$

Proof. This follows from a straightforward dualisation of the result of Lemma 3.6. □

We can now give an alternative and intrinsic proof of the following theorem.

Theorem 3.8.

$$\mathcal{L}_Z A_Z = -A_Z^2 - \bar{\Phi}.$$

Proof. From Lemma 3.6

$$\bar{\Phi}(X) = -\bar{\nabla}(A_Z(X)) + A_Z([Z, X]).$$

Expanding the first term on the right-hand side gives

$$\bar{\Phi}(X) = -\bar{\nabla} A_Z(X) - A_Z(\bar{\nabla} X) + A_Z([Z, X]).$$

Now using Proposition 3.4 and the linearity of A_Z,

$$\bar{\Phi}(X) = -\mathcal{L}_Z A_Z(X) - A_Z(\bar{\nabla} X - [Z, X]) = -\mathcal{L}_Z A_Z(X) - A_Z^2(X).$$ □

Notice that this proof relies (through Lemma 3.6) on the result $P_V \circ \mathcal{L}_\Gamma P_V = -\Phi$ of Proposition 3.1, and that although it is not established here, $\mathcal{L}_Z A_Z = -A_Z^2 - \bar{\Phi}$ is the pullback of the global equation $P_V \circ \mathcal{L}_\Gamma P_V = -\Phi$. In the geodesic case [3] the propagation equation for A_Z is found via the pullback of the global equation, except that in that paper Φ is replaced by the curvature of the linear connection. A similar method is used in [7].

The dual version of Theorem 3.8 is obtained by direct dualisation or by using Lemma 3.7.

Corollary 3.9.

$$\mathcal{L}_Z A_Z^* = -A_Z^{*2} - \bar{\Phi}^*.$$

The last part of the generalisation of the geometry of geodesic congruences to be presented here is the extension of the Jacobi tensor concept described in Section 2. (This topic is not covered in [7] and we refer the reader to [2,10,15,16] for earlier appearances of this generalisation.) For our purposes we do not distinguish the Jacobi (or linear variational) equation from its classical adjoint by name.

Definition 3.10. Any tensor field J on $\mathbb{R} \times M$ satisfying

$$\bar{\nabla}^2 J = -\bar{\Phi} \circ J \quad \text{or} \quad \bar{\nabla}^2 J = -\bar{\Phi}^* \circ J = -J \circ \bar{\Phi}$$

is called a *Jacobi tensor*.

Proposition 3.11. *Any tensor field J on $\mathbb{R} \times M$ satisfying*

$$\bar{\nabla} J = A_Z \circ J \quad \textit{or} \quad \bar{\nabla} J = A_Z^* \circ J$$

is a Jacobi tensor.

Proof. We prove only the $\bar{\nabla} J = A_Z \circ J$ part:

$$\begin{aligned} \bar{\nabla}^2 J = \bar{\nabla}(A_Z \circ J) &= \bar{\nabla} A_Z \circ J + A_Z \circ \bar{\nabla} J \\ &= (-A_Z^2 - \bar{\Phi}) \circ J + A_Z^2 \circ J \quad \text{using Theorem 3.8} \\ &= -\bar{\Phi} \circ J \end{aligned}$$

as required. □

The symmetries and the adjoint symmetries of Z (see [15]) are examples of Jacobi fields satisfying $\bar{\nabla} J = A_Z \circ J$ and $\bar{\nabla} J = A_Z^* \circ J$, respectively.

4. A linear connection on E

In [8], Massa and Pagani introduced a linear connection on E by imposing some natural requirements. If we denote their connection by $\hat{\nabla}$, these are that the covariant differentials $\hat{\nabla}\, \mathrm{d}t$, $\hat{\nabla} S$, and $\hat{\nabla} \Gamma$ are all zero and that the vertical sub-bundle is flat. They do, in fact, produce a shape map associated with $\hat{\nabla}$ and Γ, although from a rather different perspective than that of our study here.

Massa and Pagani's ideas have been acknowledged but not been widely adopted partly because of a trend in the literature to work on a certain pullback bundle over E. We will describe the work of Crampin et al. [2] in developing a linear connection in this context and show that once the treatment of the nonautonomous case is freed from some misleading features of the autonomous one, Massa and Pagani's connection coincides with a straightforward modification of that of Crampin et al. (Since we completed this part of the work, Mestdag and Sarlet [9,14] have modified the connection of Crampin et al. [2] to one which lifts to that of Massa and Pagani.)

There is also the matter of the utility of the whole pullback bundle approach given Massa and Pagani's independence of it, however, we believe it has some computational and conceptual advantages which we will attempt to utilise.

So now we introduce vector fields and forms *along the projection* $\pi_1^0 : E \to \mathbb{R} \times M$. We follow [16]. Vector fields along π_1^0 are sections of the pullback bundle $\pi_1^{0*}(T(\mathbb{R} \times M))$ over E. $\mathfrak{X}(\pi_1^0)$ denotes the $C^\infty(E)$ module of such vector fields. Similarly, $\bigwedge(\pi_1^0)$ denotes the graded algebra of scalar-valued forms along π_1^0 and $V(\pi_1^0)$ denotes the $\bigwedge(\pi_1^0)$-module of vector-valued forms along π_1^0. *Basic vector fields and* 1-*forms* along π_1^0 are elements of $\mathfrak{X}(\mathbb{R} \times M)$ and $\mathfrak{X}^*(\mathbb{R} \times M)$, respectively, identified with vector fields and forms along π_1^0 by composition with π_1^0. Using this device tensor fields along the projection can be expressed as tensor products of basic vector fields and 1-forms with coefficients in $C^\infty(E)$. The canonical vector field along π_1^0 is

$$\mathbf{T} = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a},$$

and the natural bases for $\mathfrak{X}(\pi_1^0)$ and $\mathfrak{X}^*(\pi_1^0)$ are then $\{\mathbf{T}, (\partial/\partial x^a)\}$ and $\{\mathrm{d}t, \theta^a\}$. The set of equivalence classes of vector fields along π_1^0 modulo $\mathbf{T}$ is denoted $\overline{\mathfrak{X}(\pi_1^0)}$ so that $\bar{X} \in \overline{\mathfrak{X}(\pi_1^0)}$ satisfies $\mathrm{d}t(\bar{X}) = 0$. Then the obvious bijection between $\overline{\mathfrak{X}(\pi_1^0)}$ and $V(E)$ provides a *vertical lift* from $\mathfrak{X}(\pi_1^0)$ to $V(E)$, given in coordinates by

$$X^V = \bar{X}^a \frac{\partial}{\partial u^a} = (X^a - u^a X^0) \frac{\partial}{\partial u^a},$$

where $X = X^0(\partial/\partial t) + X^a(\partial/\partial x^a)$.

On the matter of horizontal lifts we part company with [16] and say that the *horizontal lift* X^H of $X \in \mathfrak{X}(\pi_1^0)$ is given by $X^H = \bar{X}^a H_a$. (There are many reasons for this: for example, it is consistent with the horizontal lift of Crampin et al. [4] and it respects the eigenvector structure of $\mathcal{L}_\Gamma S$, for this reason it is also known as the *strong* horizontal lift, see [5].) Finally, we can *lift along* Γ by $X^\Gamma := \mathrm{d}t(X)\Gamma$ for any $X \in \mathfrak{X}(\pi_1^0)$ (so that $\mathbf{T}^\Gamma = \Gamma$). Then any vector field $W \in \mathfrak{X}(E)$ can be decomposed as

$$W = (W_\Gamma)^\Gamma + (W_H)^H + (W_V)^V$$

for unique $W_\Gamma \in \mathrm{Sp}\{\mathbf{T}\}$, $W_H \in \mathfrak{X}(\pi_1^0)$ with $W_H(t) = W(t)$ and $W_V \in \overline{\mathfrak{X}(\pi_1^0)}$. This decomposition is the main aim of the lifting exercise. In coordinates,

$$W_\Gamma = \mathrm{d}t(W)\mathbf{T}, \qquad W_H = \mathrm{d}t(W)\frac{\partial}{\partial t} + \mathrm{d}x^a(W)\frac{\partial}{\partial x^a} = \mathrm{d}t(W)\mathbf{T} + \theta^a(W)\frac{\partial}{\partial x^a},$$
$$W_V = \psi^a(W)\frac{\partial}{\partial x^a}.$$

The *dynamical covariant derivative* ∇ and the Jacobi endomorphism, Φ, are then defined as objects along the projection through the following commutation relations on E:

$$[\Gamma, X^V] = -X^H + (\nabla X)^V, \qquad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V. \tag{4.1}$$

In coordinates $\Phi = \Phi^a_b(\partial/\partial x^a) \otimes \theta^b$ (we make no notational distinction between the Jacobi endomorphism in this context and in that of the previous section). We extend ∇ to act on forms by setting $\nabla(F) := \Gamma(F)$ for $F \in \bigwedge^0(\pi_1^0)$; then it can be shown that $\nabla(\langle X, \alpha\rangle) = \langle\nabla X, \alpha\rangle + \langle X, \nabla\alpha\rangle$ and so ∇ can be extended to tensor fields along π_1^0 in the usual way. $\nabla\mathbf{T} = 0$ and, in coordinates,

$$\nabla\theta^a = -\Gamma^a_b\theta^b, \qquad \nabla\,\mathrm{d}t = 0, \qquad \nabla\frac{\partial}{\partial x^a} = \Gamma^b_a\frac{\partial}{\partial x^b}.$$

Massa and Pagani [8], Byrnes [1] and Crampin et al. [2] have separately proposed various linear connections on E induced by an SODE Γ. They all use the dynamical covariant derivative ∇ to determine derivatives *along* Γ, but differ in the derivatives *of* Γ. This is essentially equivalent to different choices of the torsion. Crampin et al. [2] firstly define a covariant derivative along π_1^0 and then induce one on E by lifting. They are rather insistent that this process produces a more natural and economical linear connection than the others, but as we will see, there remains quite a deal of freedom even in defining the covariant derivative along the projection. We will show that Massa and Pagani's linear connection on E is induced by one along the projection which is quite transparent if natural projectors P_Γ, P_H and P_V are used.

The covariant derivative, $D_Y U$, along π_1^0 in [2] is defined for each $Y \in \mathfrak{X}(E)$ and $U \in \mathfrak{X}(\pi_1^0)$ as follows:

$$D_Y U := [P_{\mathcal{H}}(Y), U^{\mathcal{V}}]_{\mathcal{V}} + [P_{\mathcal{V}}(Y), U^{\mathcal{H}}]_{\mathcal{H}} + P_{\mathcal{H}}(Y)\langle U, \mathrm{d}t\rangle\mathbf{T},$$

where $\mathcal{H}$'s and $\mathcal{V}$'s correspond to the authors' alternative splitting of $\mathfrak{X}(E)$ in which Γ and the horizontal distribution are lumped together. Taking D as a model we define a covariant derivative $\hat{D}$ along the projection as follows.

Proposition 4.1. *For each $Y \in \mathfrak{X}(E)$, $U \in \mathfrak{X}(\pi_1^0)$ and $f \in C^\infty(E)$,*

$$\hat{D}_Y U := [P_H(Y), U^V]_V + [P_\Gamma(Y), U^V]_V + [P_V(Y), U^H]_H + Y(U(t))\mathbf{T},$$
$$\hat{D}_Y(f) := Y(f)$$

is a covariant derivative.

Proof. Since $(P_H(Y))_V = (P_\Gamma(Y))_V = (P_V(Y))_H = 0$, it is clear that $\hat{D}_{fY}U = f\hat{D}_Y U$. Secondly, one has

$$\hat{D}_Y(fU) = f\hat{D}_Y U + [P_H(Y)(f) + P_\Gamma(Y)(f)]U^V_V + P_V(Y)(f)U^H_H + U(t)Y(f)\mathbf{T},$$

and since $U_V^V = U_H^H = \theta^a(U)(\partial/\partial x^a)$ the right-hand side becomes

$$f\hat{D}_Y U + Y(f)\theta^a(U)\frac{\partial}{\partial x^a} + U(t)Y(f)\mathbf{T}$$

giving

$$\hat{D}_Y(fU) = f\hat{D}_Y U + Y(f)U. \qquad \square$$

Proposition 4.2. *The components of $\hat{D}$ are as follows:*

$$\hat{D}_\Gamma \mathbf{T} = 0, \qquad \hat{D}_{H_a}\mathbf{T} = 0, \qquad \hat{D}_{V_a}\mathbf{T} = 0, \qquad \hat{D}_\Gamma \frac{\partial}{\partial x^a} = \Gamma_a^b \frac{\partial}{\partial x^b},$$
$$\hat{D}_{H_b}\frac{\partial}{\partial x^a} = \frac{\partial \Gamma_a^c}{\partial u^b}\frac{\partial}{\partial x^c}, \qquad \hat{D}_{V_b}\frac{\partial}{\partial x^a} = 0.$$

Proof. Since $P_H(\Gamma) = P_V(\Gamma) = 0$, $\mathbf{T}^V = 0$ and $\Gamma(\mathbf{T}(t)) = \Gamma(1) = 0$ every term in $\hat{D}_\Gamma \mathbf{T}$ vanishes. Similarly, for $\hat{D}_{H_a}\mathbf{T}$ and $\hat{D}_{V_a}\mathbf{T}$. Otherwise

$$\hat{D}_\Gamma \frac{\partial}{\partial x^a} = [\Gamma, V_a] = (-H_a + \Gamma_a^c V_c)_V = \Gamma_a^c \frac{\partial}{\partial x^c},$$
$$\hat{D}_{H_b}\frac{\partial}{\partial x^a} = [H_b, V_a]_V = \left(-\frac{\partial \Gamma_b^c}{\partial u^a} V_c\right)_V = -\frac{\partial \Gamma_b^c}{\partial u^a}\frac{\partial}{\partial x^c},$$
$$\hat{D}_{V_b}\frac{\partial}{\partial x^a} = [V_b, H_a]_H = \left(\frac{\partial \Gamma_a^c}{\partial u^b} V_c\right)_H = 0. \qquad \square$$

In fact, there is a simple relationship between D and $\hat{D}$, namely

$$\hat{D}_Y U = D_Y U - U(t)(P_V(Y))_V.$$

We now use $\hat{D}$ to define a linear connection $\hat{\nabla}$ on E (it is not an accident that we denote Massa and Pagani's derivative with the same symbol) in the manner of Crampin et al. [2].

Proposition 4.3.

$$\hat{\nabla}_Y X := (\hat{D}_Y X_\Gamma)^\Gamma + (\hat{D}_Y X_H)^H + (\hat{D}_Y X_V)^V, \qquad \hat{\nabla}_Y(f) := Y(f)$$

for all $Y, X \in \mathfrak{X}(E)$ and $f \in C^\infty(E)$ is a linear covariant derivative.

Proof. Use the linearity of all the lifts and projections, and the fact that $\hat{D}$ is a covariant derivative. $\square$

This linear connection is identical to that of Massa and Pagani [8] as can be verified by calculating the covariant differentials of S, $\mathrm{d}t$ and Γ along with $\hat{\nabla}_{V_a} X$ or directly from the components below

$$\hat{\nabla}_\Gamma \Gamma = 0, \qquad \hat{\nabla}_\Gamma H_a = \Gamma_a^b H_b, \qquad \hat{\nabla}_\Gamma V_a = \Gamma_a^b V_b, \qquad \hat{\nabla}_{H_a}\Gamma = 0,$$
$$\hat{\nabla}_{H_a} H_b = \frac{\partial \Gamma_a^c}{\partial u^b} H_c, \quad \hat{\nabla}_{H_a} V_b = \frac{\partial \Gamma_a^c}{\partial u^b} V_c, \quad \hat{\nabla}_{V_a}\Gamma = 0, \quad \hat{\nabla}_{V_a} H_b = 0, \quad \hat{\nabla}_{V_a} V_b = 0.$$

A key feature of $\hat{\nabla}$ for us is that $\hat{\nabla}_X \Gamma = 0$ for all $X \in \mathfrak{X}(\pi_1^0)$. It is also worth noting the following important facts which follow from Propositions 4.2 and 4.3 and noting that $\hat{D}_\Gamma = \nabla$.

Corollary 4.4. *Let* $X, Y \in \mathfrak{X}(\pi_1^0)$. *Then*

$$\begin{aligned} &\hat{\nabla}_\Gamma X^V = (\nabla X)^V, \qquad \hat{\nabla}_\Gamma X^H = (\nabla X)^H, \qquad \hat{\nabla}_{Y^H} X^H = (\hat{D}_{Y^H} X)^H, \\ &\hat{\nabla}_{Y^H} X^V = (\hat{D}_{Y^H} X)^V, \qquad \hat{\nabla}_{Y^V} X^H = (\hat{D}_{Y^V} X)^H, \\ &\hat{\nabla}_{Y^V} X^V = (\hat{D}_{Y^V} X)^V, \qquad \hat{\nabla}_{Y^H} X^\Gamma = Y^H(\mathrm{d}t(X))\Gamma, \qquad \hat{\nabla}_{Y^V} X^\Gamma = Y^V(\mathrm{d}t(X))\Gamma. \end{aligned}$$

5. The shape map A_Γ

We now have evolution space E equipped with a linear connection $\hat{\nabla}$ and torsion

$$\hat{T}(X, Y) := \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y]. \tag{5.1}$$

As a consequence of Theorem 1.2 we immediately have, for any SODE:

$$A_X(Y) = \hat{\nabla}_Y X + \hat{T}(X, Y) = \hat{\nabla}_X Y - [X, Y], \tag{5.2}$$

and because $\hat{\nabla}_X \Gamma = 0$ for all $X \in \mathfrak{X}(E)$,

$$A_\Gamma(X) = \hat{T}(\Gamma, X) = \hat{\nabla}_\Gamma X - [\Gamma, X]. \tag{5.3}$$

Note the similarity of this last expression to the equation in part (ii) of Definition 3.3 of $\bar{\nabla}$, although that definition was modelled on the conventional identity for zero torsion. We exhibit an alternative, coordinate-free, expression for A_Γ.

Proposition 5.1.

$$A_\Gamma = -P_V \circ \mathcal{L}_\Gamma P_H - P_H \circ \mathcal{L}_\Gamma P_V.$$

Proof. The components of A_Γ are

$$\begin{aligned} &A_\Gamma(\Gamma) = \hat{T}(\Gamma, \Gamma) = 0, \qquad A_\Gamma(H_a) = \hat{T}(\Gamma, H_a) = -\Phi_a^b V_b, \\ &A_\Gamma(V_a) = \hat{T}(\Gamma, V_a) = H_a, \end{aligned}$$

so that relative to the usual basis $\{\Gamma, H_a, V_a\}$

$$A_\Gamma = -\Phi_b^a V_a \otimes \theta^b + H_a \otimes \psi^a. \tag{5.4}$$

It is easy to see that $P_H \circ \mathcal{L}_\Gamma P_V = P_H \circ (\mathcal{L}_\Gamma \circ P_V - P_V \circ \mathcal{L}_\Gamma) = P_H \circ \mathcal{L}_\Gamma \circ P_V$. Furthermore, since $P_H \circ \mathcal{L}_\Gamma \circ P_V$ is linear (over $C^\infty(E)$) so is $P_H \circ \mathcal{L}_\Gamma P_V$. It also follows that $P_H \circ \mathcal{L}_\Gamma P_V(H_a) = P_H \circ \mathcal{L}_\Gamma P_V(\Gamma) = 0$. Taking the horizontal part of Eq. (3.8)

$$\begin{aligned} H_a &= -P_H(\mathcal{L}_\Gamma V_a) = -P_H(\mathcal{L}_\Gamma(P_V(V_a))) = -P_H(\mathcal{L}_\Gamma P_V(V_a)) \\ &= -P_H \circ \mathcal{L}_\Gamma P_V(V_a). \end{aligned}$$

It follows that $-P_H \circ \mathcal{L}_\Gamma P_V$ is the type $(1, 1)$ tensor field which is zero on H_a and Γ and sends V_a to H_a. Hence

$$A_\Gamma = -\Phi - P_H \circ \mathcal{L}_\Gamma P_V = -P_V \circ \mathcal{L}_\Gamma P_H - P_H \circ \mathcal{L}_\Gamma P_V,$$

where in the last step we used Proposition 3.1 to replace Φ. □

We have the following immediate corollary (following from (5.4)).

Corollary 5.2.

$$A_\Gamma(X) = -\Phi(X_H)^V + (X_V)^H. \tag{5.5}$$

It is worth remarking the importance of Eqs. (5.3) and (5.5) in relating the three operations $\mathcal{L}_\Gamma$, $\hat{\nabla}_\Gamma$ and A_Γ, and also that Eq. (4.1) are equivalent to them

$$[\Gamma, X^V] = \hat{\nabla}_\Gamma X^V - X^H, \qquad [\Gamma, X^H] = \hat{\nabla}_\Gamma X^H + \Phi(X)^V.$$

Now we compare A_Γ to A_Z. Tangent spaces to $\mathbb{R}\times M$ are spanned by $\{Z, (\partial/\partial x^a)\}$, therefore the tangent spaces to the image of the section, $\sigma_Z(\mathbb{R} \times M)$, are spanned by $\{\Gamma, \Sigma_a\}$ since $\Gamma \stackrel{*}{=} \sigma_{Z*} Z$ and we define $\Sigma_a := \sigma_{Z*}(\partial/\partial x^a)$. Now

$$\Sigma_a := \sigma_{Z*}\frac{\partial}{\partial x^a} \stackrel{*}{=} \frac{\partial}{\partial x^a} + \frac{\partial Z^b}{\partial x^a}\frac{\partial}{\partial u^b} \stackrel{*}{=} H_a + \left(\bar{\Gamma}_a^b + \frac{\partial Z^b}{\partial x^a}\right) V_b,$$

so clearly vectors tangent to the image of the section will be annihilated by the n-annihilating forms

$$\Lambda^a :\stackrel{*}{=} \psi^a - \left(\bar{\Gamma}_b^a + \frac{\partial Z^a}{\partial x^b}\right)\theta^b.$$

The corresponding bases for $T_{\sigma_Z(p)}E$ and its dual are then $\{\Gamma, \Sigma_a, V_a\}$ and $\{\mathrm{d}t, \theta^a, \Lambda^a\}$. So we have

$$\begin{aligned} A_\Gamma \stackrel{*}{=} &(A_Z)_b^a \Sigma_a \otimes \theta^b + (-\bar{\Phi}_b^a - (A_Z)_c^a (A_Z)_b^c) V_a \otimes \theta^b \\ &- (A_Z)_b^a V_a \otimes \Lambda^b + \Sigma_a \otimes \Lambda^a, \end{aligned} \tag{5.6}$$

where we used $(A_Z)_b^a = (\bar{\Gamma}_b^a + (\partial Z^a/\partial x^b))$, see Eq. (3.12). Furthermore, Theorem 3.8 tells us $\bar{\nabla} A_Z = \mathcal{L}_Z A_Z = -\bar{\Phi} - A_Z^2$, hence this becomes

$$A_\Gamma \stackrel{*}{=} (A_Z)_b^a \Sigma_a \otimes \theta^b + (\bar{\nabla} A_Z)_b^a V_a \otimes \theta^b - (A_Z)_b^a V_a \otimes \Lambda^b + \Sigma_a \otimes \Lambda^a. \tag{5.7}$$

Now we turn to the spectral analysis of A_Γ. First of all it is clear from Eqs. (5.4) and (5.5) that $\mathrm{tr}(A_\Gamma) = 0$ and that the eigenvectors X belonging to the zero eigenfunction of A_Γ lie in $\mathrm{Sp}\{\Gamma, H_a\}$ with $\Phi(X_H) = 0$. So we will suppose that if $A_\Gamma(X) = \lambda X$ then $P_\Gamma(X) = 0$.

Theorem 5.3. *Suppose that $X \in \mathfrak{X}(E)$ with $P_\Gamma(X) = 0$ and $\lambda \in C^\infty(E)$. Then*

$$A_\Gamma(X) = \lambda X \Leftrightarrow \Phi(X_H) = -\lambda^2 X_H \quad \textit{and} \quad P_V(X) = \lambda P_H(X).$$

Proof.

$$\begin{aligned} A_\Gamma(X) = \lambda X &\Leftrightarrow -\Phi(X_H)^V + (X_V)^H = \lambda((X_H)^H + (X_V)^V) \\ &\Leftrightarrow -\Phi(X_H) = X_V \quad \text{and} \quad X_V = \lambda X_H \\ &\Leftrightarrow \Phi(X_H) = -\lambda^2 X_H \quad \text{and} \quad X_V = \lambda X_H. \end{aligned}$$ □

In the Riemannian case the $(n \times n)$ matrix representation of Φ, $\mathbf{\Phi}$, is symmetric, but for an arbitrary SODE this is not generally true. However, the cases where $\mathbf{\Phi}$ has real eigenvalues can be geometrically characterised.

Corollary 5.4. *The eigenvalues of* $\mathbf{\Phi}$ *are real if and only if the eigenvalues of* A_Γ *are real pairs of opposite sign and/or pure imaginary* (*pairs of opposite sign*).

We now investigate those sections σ_Z invariant under A_Γ. Notice that the restriction of A_Γ to $T_{\sigma_z(p)}E$ consists of the first two terms on the right-hand side of (5.7), indicating that $\bar\nabla A_Z$ measures the failure of A_Γ to preserve these tangent spaces. (Explicitly $A_\Gamma(\sigma_{Z*}X) = \sigma_{Z*}(A_Z(X)) + (\bar\nabla A_Z)(X)^V$.) Hence σ_Z is invariant under A_Γ (in the sense that the tangent spaces to the image of the section are invariant subspaces of A_Γ) if and only if $\bar\nabla A_Z = 0$. On the other hand, from (5.6), a direction $\sigma_{Z*}X$ tangent to the image of the section is invariant under A_Γ if and only if $A_Z(X) = \lambda X$ and $\bar\Phi(X) = -\lambda^2 X$ for some local function λ on $\mathbb{R} \times M$. As a consequence a section σ_Z is strictly invariant under A_Γ if and only if A_Z is a multiple of the identity at each point and $\bar\Phi = -A_Z^2$.

6. Jacobi fields on *E*

6.1. *The Raychaudhuri equation for* A_Γ

In the geodesic case [3], Crampin and Prince work on the tangent bundle *TM* of a differentiable manifold *M* with linear connection. There they find a propagation equation for A_Z using the curvature $R_Z = R(\cdot, Z)Z$. However, in the present situation an attempt to use curvature

$$\hat R(X, Y)Z := \hat\nabla_X \hat\nabla_Y Z - \hat\nabla_Y \hat\nabla_X Z - \hat\nabla_{[X,Y]} Z$$

to find a propagation equation for A_Γ in the same way fails because $\hat R(X, \Gamma)\Gamma \equiv 0$. We proceed by directly differentiating A_Γ.

Lemma 6.1.

$$\hat\nabla_\Gamma A_\Gamma(X) = \hat\nabla_\Gamma \hat T(\Gamma, X).$$

Proof.

$$\begin{aligned} \hat\nabla_\Gamma A_\Gamma(X) &= \hat\nabla_\Gamma(\hat T(\Gamma, X)) - \hat T(\Gamma, \hat\nabla_\Gamma X) \\ &= \{\hat\nabla_\Gamma \hat T(\Gamma, X) + \hat T(\Gamma, \hat\nabla_\Gamma X)\} - \hat T(\Gamma, \hat\nabla_\Gamma X) = (\hat\nabla_\Gamma \hat T)(\Gamma, X). \end{aligned}$$ □

Lemma 6.2.

$$\mathcal{L}_\Gamma A_\Gamma = \hat{\nabla}_\Gamma A_\Gamma .$$

Proof. Expand and use Eq. (5.3).

$$\begin{aligned}(\mathcal{L}_\Gamma A_\Gamma)(X) &= \mathcal{L}_\Gamma(A_\Gamma(X)) - A_\Gamma(\mathcal{L}_\Gamma X) = [\Gamma, A_\Gamma(X)] - A_\Gamma([\Gamma, X]) \\ &= \hat{\nabla}_\Gamma(A_\Gamma(X)) - A_\Gamma(A_\Gamma(X)) - A_\Gamma(\hat{\nabla}_\Gamma X - A_\Gamma(X)) \\ &= \hat{\nabla}_\Gamma(A_\Gamma(X)) - A_\Gamma(\hat{\nabla}_\Gamma X) = \hat{\nabla}_\Gamma A_\Gamma(X). \qquad \square\end{aligned}$$

Recall that $\mathrm{tr}(A_\Gamma) = 0$ so that $\mathrm{tr}(\mathcal{L}_\Gamma A_\Gamma) = 0$ and so a generalised Raychaudhuri equation will be trivial, however, we can obtain a propagation equation for A_Γ along Γ directly from the above lemmas which we will call the generalised Raychaudhuri equation for Γ. We give it in two forms.

Theorem 6.3.

$$\mathcal{L}_\Gamma A_\Gamma(X) = (\hat{\nabla}_\Gamma \hat{T})(\Gamma, X).$$

Equivalently,

$$\mathcal{L}_\Gamma A_\Gamma = -\hat{\nabla}_\Gamma \Phi .$$

Proof. The first follows immediately from the lemmas. It follows from Lemma 6.2 and Proposition 5.1 or its corollary that

$$\mathcal{L}_\Gamma A_\Gamma = \hat{\nabla}_\Gamma A_\Gamma = -\hat{\nabla}_\Gamma \Phi . \qquad \square$$

In coordinates,

$$\mathcal{L}_\Gamma A_\Gamma = (\Gamma_a^b \Phi_b^c - \Gamma_b^c \Phi_a^b - \Gamma(\Phi_a^c)) V_c \otimes \theta^a .$$

The generalised Jacobi equation deals with Lie-dragged vector fields along integral curves of Γ.

Theorem 6.4 (The Jacobi equation for $\hat{\nabla}$). *Let X satisfy $A_\Gamma(X) = \hat{\nabla}_\Gamma X$ (equivalently $[\Gamma, X] = 0$). Then*

$$\hat{\nabla}_\Gamma^2 X = (\mathcal{L}_\Gamma A_\Gamma + A_\Gamma^2)(X). \tag{6.1}$$

Proof.

$$\begin{aligned}\hat{\nabla}_\Gamma^2 X &= \hat{\nabla}_\Gamma((A_\Gamma(X))) = (\hat{\nabla}_\Gamma A_\Gamma)(X) + A_\Gamma(\hat{\nabla}_\Gamma X) \\ &= (\mathcal{L}_\Gamma A_\Gamma)(X) + A_\Gamma(A_\Gamma(X)) = (\mathcal{L}_\Gamma A_\Gamma + A_\Gamma^2)(X). \qquad \square\end{aligned}$$

We claim that Eq. (6.1) is a generalised geodesic deviation equation for an arbitrary SODE. The following corollary shows that the horizontal component of Eq. (6.1) is the generalised Jacobi equation given in [2] (and that the vertical component is the $\hat{D}_\Gamma$ derivative of the horizontal one). This is the same as the generalisation given in [10]. First we need the following lemma.

Lemma 6.5. *For any* $X \in \mathfrak{X}(E)$

$$[\Gamma, X] = 0 \Leftrightarrow \hat{D}_\Gamma X_\Gamma = 0, \qquad \hat{D}_\Gamma X_H = X_V, \qquad \hat{D}_\Gamma X_V = -\Phi(X_H).$$

Proof. Use $[\Gamma, X] = 0 \Leftrightarrow A_\Gamma(X) = \hat{\nabla}_\Gamma X$, Proposition 4.3 and (5.5). □

Corollary 6.6. *Let* X *satisfy* $[\Gamma, X] = 0$. *Then the horizontal and vertical components of* (6.1) *are, respectively,*

$$\hat{D}_\Gamma^2 X_H = -\Phi(X_H), \qquad \hat{D}_\Gamma^2 X_V = -\hat{D}_\Gamma(\Phi(X_H)).$$

Proof. A straightforward application of the definition of $\hat{\nabla}$ from Proposition 4.3 gives, for arbitrary X,

$$\hat{\nabla}_\Gamma(\hat{\nabla}_\Gamma X) = (\hat{D}_\Gamma^2 X_\Gamma)^\Gamma + (\hat{D}_\Gamma^2 X_H)^H + (\hat{D}_\Gamma^2 X_V)^V.$$

Eq. (5.5) gives

$$A_\Gamma^2(X) = -(\Phi(X_H))^H - \Phi(X_V)^V.$$

Applying Lemma 6.2, (5.5) and Lemma 6.5, we have

$$\begin{aligned}(\mathcal{L}_\Gamma A_\Gamma)(X) &= (\hat{\nabla}_\Gamma A_\Gamma)(X) = \hat{\nabla}_\Gamma(A_\Gamma(X)) - A_\Gamma(\hat{\nabla}_\Gamma X)\\ &= (\hat{D}_\Gamma X_V)^H - \hat{D}_\Gamma(\Phi(X_H))^V - A_\Gamma((\hat{D}_\Gamma X_H)^H) - A_\Gamma((\hat{D}_\Gamma X_V)^V)\\ &= (\hat{D}_\Gamma X_V)^H - \hat{D}_\Gamma(\Phi(X_H))^V + \Phi(\hat{D}_\Gamma X_H)^V - (\hat{D}_\Gamma X_V)^H\\ &= -\hat{D}_\Gamma(\Phi(X_H))^V + \Phi(\hat{D}_\Gamma X_H)^V = -\hat{D}_\Gamma(\Phi(X_H))^V + \Phi(X_V)^V.\end{aligned}$$

Combining these three expressions in Eq. (6.1) and equating horizontal and vertical parts completes the proof. □

Remark 6.7.

1. The vertical component of our generalised Jacobi equation is the $\hat{D}_\Gamma$ derivative of the horizontal one because, by Lemma 6.5,

$$\hat{D}_\Gamma^2 X_H = -\Phi(X_H) \;\; \Rightarrow \;\; \hat{D}_\Gamma X_V = -\Phi(X_H).$$

2. From the proof, for any $X \in \mathfrak{X}(E)$,

$$\begin{aligned}(\mathcal{L}_\Gamma A_\Gamma + A_\Gamma^2)(X) &= -\Phi(X_H)^H - \Phi(X_V)^V - \hat{D}_\Gamma(\Phi(X_H))^V + \Phi(\hat{D}_\Gamma X_H)^V\\ &= -\Phi(X_H)^H - \Phi(X_V)^V - (\hat{D}_\Gamma \Phi)(X_H)^V,\end{aligned}$$

which should be compared with the result of Theorem 3.8:

$$\mathcal{L}_Z A_Z + A_Z^2 = \bar{\Phi}.$$

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Appendix A. Tensor character of A_Z

Our analysis begins with a given SODE, and parameterisation. Hence for us the basic geometrical object is, by construction, a product manifold—the graph space of the configuration space of the system. Therefore, we consider an object to be *tensorial* if its coordinate representation is unchanged under coordinate transformations which preserve the first factor of the graph space $\mathbb{R} \times M$. The object of this section is to establish the tensor character of A_Z.

We introduce new coordinates $(\hat{t}, \hat{x}^a)$, indicated by an overhat, on $\mathbb{R} \times M$ such that coordinate transformation respects the product structure of the graph space (i.e. preserves projection onto the $\mathbb{R}$ factor). The new coordinates $(\hat{t}, \hat{x}^a)$ depend on (t, x^a) by

$$\hat{t} = t, \qquad \hat{x}^a = \hat{X}^a(t, x^b) \tag{A.1a}$$

with inverse

$$t = \hat{t}, \qquad x^a = X^a(\hat{t}, \hat{x}^b). \tag{A.1b}$$

Change of coordinate bases on $\mathbb{R} \times M$ are given by the following equations:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{x}^b}{\partial t}\frac{\partial}{\partial \hat{x}^b}, \qquad \frac{\partial}{\partial x^a} = \frac{\partial \hat{x}^b}{\partial x^a}\frac{\partial}{\partial \hat{x}^b}, \tag{A.2a}$$

$$\mathrm{d}t = \mathrm{d}\hat{t}, \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial \hat{t}}\,\mathrm{d}\hat{t} + \frac{\partial x^a}{\partial \hat{x}^b}\,\mathrm{d}\hat{x}^b. \tag{A.2b}$$

Change of basis formulae in the reverse direction are obtained by interchanging the roles of the hatted and unhatted coordinates. We indicate components of tensor fields with respect to the new coordinate bases in the obvious way. Taking Z, for example,

$$Z = \mathrm{d}\hat{t}(Z)\frac{\partial}{\partial \hat{t}} + \hat{Z}^a\frac{\partial}{\partial \hat{x}^a},$$

where $\hat{Z}^a := \mathrm{d}\hat{x}^a(Z)$. On the other hand, we define $\hat{\bar{\theta}}^a := \mathrm{d}\hat{x}^a - \hat{Z}^a\,\mathrm{d}\hat{t}$.

Since $Z(\hat{t}) = Z(t) = 1$ the transformation (A.1a) and (A.1b) preserves the coordinate representation of Z, i.e.

$$Z = \frac{\partial}{\partial t} + Z^a\frac{\partial}{\partial x^a} = \frac{\partial}{\partial \hat{t}} + \hat{Z}^b\frac{\partial}{\partial \hat{x}^b}.$$

A similar calculation shows

$$\hat{\bar{\theta}}^a = \frac{\partial \hat{x}^a}{\partial x^b}\bar{\theta}^b. \tag{A.3}$$

The coordinate transformation (A.1a) and (A.1b) induces a transformation of the adapted coordinate chart (t, x^a, u^a) on E as follows:

$$\hat{t} = t, \qquad \hat{x}^a = \hat{X}^a(t, x^b), \qquad \hat{u}^a = \hat{U}^a(t, x^b, u^b) \tag{A.4a}$$

with inverse

$$t = \hat{t}, \qquad x^a = X^a(\hat{t}, \hat{x}^b), \qquad u^a = U^a(\hat{t}, \hat{x}^b, \hat{u}^b). \tag{A.4b}$$

The transformation of coordinate bases on E is given by

$$\begin{aligned}
&\frac{\partial}{\partial t} = \frac{\partial}{\partial \hat{t}} + \frac{\partial \hat{x}^b}{\partial t}\frac{\partial}{\partial \hat{x}^b} + \left(\frac{\partial^2 \hat{x}^b}{\partial t^2} + \frac{\partial^2 \hat{x}^b}{\partial t \partial x^a} u^a\right)\frac{\partial}{\partial \hat{u}^b},\\
&\frac{\partial}{\partial x^a} = \frac{\partial \hat{x}^b}{\partial x^a}\frac{\partial}{\partial \hat{x}^b} + \left(\frac{\partial^2 \hat{x}^b}{\partial x^a \partial t} + \frac{\partial^2 \hat{x}^b}{\partial (x^a)^2} u^a\right)\frac{\partial}{\partial \hat{u}^b}, \qquad \frac{\partial}{\partial u^a} = \frac{\partial \hat{u}^b}{\partial u^a}\frac{\partial}{\partial \hat{u}^b} = \frac{\partial \hat{x}^b}{\partial x^a}\frac{\partial}{\partial \hat{u}^b},\\
&\mathrm{d}t = \mathrm{d}\hat{t}, \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial \hat{t}}\,\mathrm{d}\hat{t} + \frac{\partial x^a}{\partial \hat{x}^b}\,\mathrm{d}\hat{x}^b,\\
&\mathrm{d}u^a = \left(\frac{\partial^2 x^a}{\partial \hat{t}^2} + \frac{\partial^2 x^a}{\partial \hat{t}\partial \hat{x}^b}\hat{u}^b\right)\mathrm{d}\hat{t} + \left(\frac{\partial^2 x^a}{\partial \hat{x}^b \partial \hat{t}} + \frac{\partial^2 x^a}{\partial (\hat{x}^b)^2}\hat{u}^b\right)\mathrm{d}\hat{x}^b + \frac{\partial x^a}{\partial \hat{x}^b}\,\mathrm{d}\hat{u}^b.
\end{aligned} \tag{A.5}$$

Again, changing bases in the reverse direction may be obtained from those above by interchanging the roles of the hatted and unhatted coordinates. Now, we have that

$$\hat{u}^b = \frac{\partial \hat{x}^b}{\partial t} + \frac{\partial \hat{x}^b}{\partial x^a} u^a.$$

We make the assumption that our SODE

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial u^a}$$

is tensorial so that the f^a's transform as accelerations, i.e.

$$\hat{f}^a := \mathrm{d}\hat{u}^a(\Gamma) = \frac{\partial^2 \hat{x}^a}{\partial t^2} + 2\frac{\partial^2 \hat{x}^a}{\partial t \partial x^b} u^b + \frac{\partial^2 \hat{x}^a}{\partial u^c \partial x^b} u^b u^c + \frac{\partial \hat{x}^a}{\partial x^b} f^b. \tag{A.6}$$

Given $Z \in \mathfrak{X}(\mathbb{R} \times M)$, we remind the reader that $\sigma_Z : \mathbb{R} \times M \to E$ is defined by $\sigma_Z(q) := (q, \pi_{0*}Z(q))$, where $\pi_0 : \mathbb{R} \times M \to M$ and $q \in \mathbb{R} \times M$. Let $p \in \sigma_Z(\mathbb{R} \times M)$, then $p = (\pi_1^0(p), \pi_{0*}Z(\pi_1^0(p)))$. By definition the coordinate functions u^a give the components of a vector tangent to M relative to the coordinate basis $\{\partial/\partial x^a\}$, clearly $u^a(p)$ will give the ath spatial component of Z, i.e. $u^a(p) = Z^a(\pi_1^0(p))$, an equation which will hold in any adapted coordinate chart containing p. Therefore one may use, without fear of confusion, the coordinate expression

$$\left.\frac{\partial f^a}{\partial u^b}\right|_{u^c = Z^c} \tag{A.7}$$

to mean evaluation is to take place on the image of the section.

The pullback of the connection coefficients, $\bar{\Gamma}^a_b$, transform as follows.

Lemma A.1.

$$\bar{\hat{\Gamma}}{}^a{}_b = -\frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial^2 \hat{x}^a}{\partial t \partial x^e} - \frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial^2 \hat{x}^a}{\partial x^c \partial x^e} Z^c + \frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial \hat{x}^a}{\partial x^d}\bar{\Gamma}^d_e.$$

Proof. Using Eq. (A.6)

$$\frac{\partial \hat{f}^a}{\partial \hat{u}^b} = \frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial \hat{f}^a}{\partial u^e} = \frac{\partial x^e}{\partial \hat{x}^b}\left(2\frac{\partial^2 \hat{x}^a}{\partial t \partial x^e} + 2\frac{\partial^2 \hat{x}^a}{\partial x^c \partial x^e}u^c + \frac{\partial \hat{x}^a}{\partial x^b}\frac{\partial f^b}{\partial u^e}\right).$$

Therefore, restricting to the image of the section one obtains

$$\begin{aligned}\bar{\hat{\Gamma}}{}^a{}_b &:= -\frac{1}{2}\frac{\partial \hat{f}^a}{\partial \hat{u}^b}\bigg|_{\hat{u}^a=\hat{Z}^a} = -\frac{1}{2}\frac{\partial x^e}{\partial \hat{x}^b}\left(2\frac{\partial^2 \hat{x}^a}{\partial t \partial x^e} + 2\frac{\partial^2 \hat{x}^a}{\partial x^c \partial x^e}Z^c + \frac{\partial \hat{x}^a}{\partial x^d}\frac{\partial f^d}{\partial u^e}\bigg|_{u^a=Z^a}\right)\\ &= -\frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial^2 \hat{x}^a}{\partial t \partial x^e} - \frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial^2 \hat{x}^a}{\partial x^c \partial x^e} Z^c + \frac{\partial x^e}{\partial \hat{x}^b}\frac{\partial \hat{x}^a}{\partial x^d}\bar{\Gamma}^d_e.\end{aligned}$$ □

The following theorem shows that given two (adapted) coordinate representations of $A_Z = \sigma_Z^* P_V$, the components of A_Z

$$A_{Z^a_b} := \frac{\partial Z^a}{\partial x^b} + \bar{\Gamma}^a_b, \qquad \hat{A}_{Z^a_b} := \frac{\partial \hat{Z}^a}{\partial \hat{x}^b} + \bar{\hat{\Gamma}}{}^a{}_b$$

transform in the right way from one coordinate picture to the other.

Theorem A.2. *The coordinate expression for A_Z remains unchanged under change of coordinates (A.1a) and (A.1b), i.e. A_Z is tensorial so that if*

$$A_Z = A_{Z^a_b}\frac{\partial}{\partial x^a} \otimes \bar{\theta}^b = \hat{A}_{Z^a_b}\frac{\partial}{\partial \hat{x}^a} \otimes \bar{\hat{\theta}}{}^b,$$

then

$$\hat{A}_{Z^c_e} = A_{Z^a_b}\frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial x^b}{\partial \hat{x}^e}.$$

Proof. For any type (1, 1) tensor field B, under change of coordinates (A.1a) and (A.1b)

$$B^a_b \frac{\partial}{\partial x^a} \otimes \bar{\theta}^b = \left(B^a_b \frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial x^b}{\partial \hat{x}^e}\right)\frac{\partial}{\partial \hat{x}^c} \otimes \bar{\hat{\theta}},$$

where we used Eqs. (A.2a)–(A.3) to change basis. Hence to prove A_Z is tensorial it remains to show that

$$\hat{A}_{Z^c_e} = A_{Z^a_b}\frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial x^b}{\partial \hat{x}^e}.$$

Now, $\hat{A}_{Z^a_b} = (\partial\hat{Z}^a/\partial\hat{x}^b) + \bar{\hat{\Gamma}}^a_b$. Consider the first term $\partial\hat{Z}^a/\partial\hat{x}^b$. Using Eq. (A.5),

$$\begin{aligned}\frac{\partial\hat{Z}^a}{\partial\hat{x}^b} &= \frac{\partial}{\partial\hat{x}^b}(\mathrm{d}\hat{x}^a(Z)) = \left(\frac{\partial x^c}{\partial\hat{x}^b}\frac{\partial}{\partial x^c}\right)\left(\frac{\partial\hat{x}^a}{\partial t}\,\mathrm{d}t(Z) + \frac{\partial\hat{x}^a}{\partial x^d}\,\mathrm{d}x^d(Z)\right)\\ &= \left(\frac{\partial x^c}{\partial\hat{x}^b}\frac{\partial}{\partial x^c}\right)\left(\frac{\partial\hat{x}^a}{\partial t}+\frac{\partial\hat{x}^a}{\partial x^d}Z^d\right) = \frac{\partial x^c}{\partial\hat{x}^b}\left(\frac{\partial^2\hat{x}^a}{\partial x^c\partial t}+\frac{\partial^2\hat{x}^a}{\partial x^c\partial x^d}Z^d+\frac{\partial\hat{x}^a}{\partial x^d}\frac{\partial Z^d}{\partial x^c}\right).\end{aligned}$$

Combining this result and lemma (A.1a) and (A.1b) gives

$$\begin{aligned}\hat{A}_{Z^a_b} &= \frac{\partial\hat{Z}^a}{\partial\hat{x}^b} + \bar{\hat{\Gamma}}^a_b = \frac{\partial x^c}{\partial\hat{x}^b}\frac{\partial\hat{x}^a}{\partial x^d}\frac{\partial Z^d}{\partial x^c} + \frac{\partial x^e}{\partial\hat{x}^b}\frac{\partial\hat{x}^a}{\partial x^d}\bar{\Gamma}^d_e = \frac{\partial x^c}{\partial\hat{x}^b}\frac{\partial\hat{x}^a}{\partial x^d}\left(\frac{\partial Z^d}{\partial x^c} + \bar{\Gamma}^d_e\right)\\ &= \frac{\partial x^c}{\partial\hat{x}^b}\frac{\partial\hat{x}^a}{\partial x^d}(A_{Z^d_c})\end{aligned}$$

as required. □

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